無限維橢圓偏微分方程對無限區間之控制問題
的應用
A Note on Applications of Infinite Dimensional Elliptic PDE to Infinite Horizon Control Problems

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摘要
本文主要在研究 Hilbert 空間上之橢圓偏微分方程式 $Lv(x) + ψ(x, v(x), \nabla v(x)G(x)) - \lambda v(x) = 0$ 的溫和解，並應用此解成為最小化無限區間之成本函數 $J(x_0, u) = \int _0 ^\infty e^{\lambda^2} g (X_t^u, u_t)ds$ 的最佳控制 $u_t \in \Gamma (X_t^u, \nabla v(X_t^u)G(X_t^u)R(X_t^u))$。

關鍵字：橢圓偏微分方程、轉移半群、前進後退隨微分方程、折現成本函數、柱形 Wiener 過程、Hamilton-Jacobi-Bell 方程。

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Abstract

In this note, we study the mild solutions to an elliptic partial differential equation $Lv(x) + \psi(x, v(x), \nabla v(x)G(x)) - \lambda v(x) = 0$ in a Hilbert space $H$. The solutions admit an optimal control $u_t \in \Gamma(X_t^u, \nabla v(X_t^u)G(X_t^u)R(X_t^u))$ and are applied to minimize the infinite horizon cost function $J(x_0, u) = E\int_0^\infty e^{-2s} g(X_s^u, u_s)ds$.

Keywords: elliptic partial differential equation, transition semigroup, forward-backward stochastic differential equations, discounted cost functional, cylindrical Wiener Process, Hamilton-Jacobi equation.

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1. Introduction.

Let us consider the following elliptic partial differential equation (PDE) in a Hilbert space $H$,

$$Lu(x) + \psi(x, u(x), \nabla u(x)G(x)) - \lambda u(x) = 0, \quad x \in H,$$

(1.1)

where the second order operator $L$ is ([2], [3], and [4]):

$$L \phi(x) = \frac{1}{2} \text{Trace}(G(x)G(x)^* \nabla^2 \phi(x)) + \langle Ax, \nabla \phi(x) \rangle + \langle F(x), \nabla \phi(x) \rangle.$$

Next, we consider that $H$ and $\Xi$ are two separable Hilbert spaces. $A$ is the generator of a strongly continuous semi-group of bounded linear operators $(e^{itA})_{t \geq 0}$ in $H$; $F$ and $G$ are functions with values in $H$ and $L(\Xi, H)$ respectively, satisfying appropriate Lipschitz conditions; $\psi$ is a function from $H \times \mathbb{R} \times \Xi^*$ to $\mathbb{R}$, and $\lambda > 0$.

When considering the problem of minimizing the discounted cost function on infinite horizon, we are concerned with the characterization of the mild solution of the stationary PDE (1.1), also named Hamilton-Jacobi equation. The corresponding stochastic differential equation of PDE (1.1) is

$$\begin{cases}
    dX_s = AX_s dt + F(X_s)dt + G(X_s)dW_t, & s \leq t, \\
    X_s = x,
\end{cases}$$

(1.2)

where $s \geq 0$, $x \in H$ and $W$ is a cylindrical Wiener process in the Hilbert space $\Xi$. The Markov process $X$ defines a transition semi-group $(P_t)$, acting on a bounded measurable functions $\varphi : H \to \mathbb{R}$, by the formula:

$$P_{t-t_0}[\varphi](x) = E_\varphi(X_t^{x,t_0}), \quad x \in H, \quad t \geq t_0 \geq 0.$$

Besides, $L$, defined on regular enough functions, is the infinitesimal generator of $(P_t)_{t \geq 0}$.

Then a bounded function $u : H \to \mathbb{R}$, Gâteaux differentiable ([3], [4], [5], and [6]), is a mild solution of (1.1), if the equality

$$u(x) = e^{-\lambda T} P_T[u](x) + \int_0^T e^{-is} P_s[\psi(., u(\cdot), \nabla u(\cdot)G(\cdot))] (x)dt$$

(1.3)

holds for all $x \in H$ and $T > 0$.

Several authors have already studied the existence and uniqueness of a mild solution of equation (1.1) in infinite dimensional spaces ([1], [2], [3], and [4]). They also showed that the
gradient of $u$ in the nonlinear term ought to have smoothing properties of the semi-group $(P_t)_{t \geq 0}$. This leads to a demand for non-degeneracy conditions on $G$ satisfying $e^{\lambda t} H \subseteq Q_t H$, $\forall t > 0$ and $|Q_t^{1/2} e^{\lambda t}| \leq c t^{-\alpha}$ where $Q_t := \int_0^t e^{s \lambda} G G^* e^{s \lambda} ds$ and $c > 0$, $\alpha \in [0, 1)$ are suitable constants.

The solution of equation (1.3) is represented by using a Markovian forward- backward system of equations

$$
\begin{align*}
\frac{dX_t}{dt} &= AX_t dt + F(X_t) dt + G(X_t) dW_t, \quad t \geq 0 \\
\frac{dY_t}{dt} &= \lambda Y_t dt - \psi(X_t, Y_t, Z_t) dt + Z_t dW_t, \quad t \geq 0 \\
X_0 &= x,
\end{align*}
$$

where the final condition for the second equation has been replaced by an appropriate growth condition [3]. The existence and uniqueness of the mild solution to equation (1.1) is proved in [2] only when $\lambda$ is large enough. However, it is inconvenient for interpretation when $\lambda$ is large. In [3], the authors employed the infinite horizon backward stochastic differential equations as (1.4) to verify that, under suitable assumptions, the mild solution to (1.1) exists and is unique for all $\lambda > 0$. They still allow $G$ to be degenerate or dependent on $x$, though they have assumed that either $G$ is non-degenerate or $G$ is constant and $A + \nabla F$ is dissipative.

The goal of this paper is to apply certain results from the elliptic partial differential equations to an optimal control problem with state equation:

$$
\begin{align*}
\frac{dX_t}{dt} &= [A X_t + F(X_t) + G(X_t) R(u) (X_t) + G(X_t) \gamma(u)] dt + G(X_t) dW_t, \\
X_0 &= x \in H,
\end{align*}
$$

where $u$ denotes the control process, taking values in a given subset of a Banach space $U$, $R$ is a mapping with values in $L(U, \Xi)$, and $\gamma$ is a bounded function with values from $U$ to $U$. Our purpose is to present the predictable control process $u$ that minimizes an infinite horizon cost functional of the form

$$
J(x, u) = \mathbb{E} \int_0^\infty e^{-\lambda s} g(X_s, u_s) ds,
$$

where $g$ is a given real bounded function and $\lambda$ is any positive number. The mild solution obtained on equation (1.1) is also the value function of the above problem (1.5) which corresponds to Hamilton-Jacobi-Bellman equation in control problem on an infinite time horizon.
It will show that the optimal control $u_t$ of (1.5) can be expressed in terms of a feedback, $u_t \in \Gamma(X^n_t, \nabla v(X^n_t)G(X^n_t)R(X^n_t))$, involving the gradient of $v(x)$ which is the solution to equation (1.1).

The structure of this note is as follows. In section 2, we introduce some important notations, concepts, hypotheses, and theorems ([2], [3], [4], and [5]) which will be used in later sections. The main result and its proof will be discussed in section 3. We will end with some applicable examples in section 4.

2. Preliminaries.

2.1. Notations.

In this section, we need to study some concepts which are useful in later parts of this note. For convenience, we follow the same notations as in [1], [2], [3], [4], and [6]. The norm of an element $x$ of a Banach space $E$ will be denoted $|x_E|$ or simply $|x|$, if no misunderstanding is possible. If $F$ is another Banach space, $L(E,F)$ indicates the space of bounded linear operators from $E$ to $F$, endowed with the usual operator norm.

The letters $\Xi$, $H$, $K$ will always denote real and separable Hilbert spaces. Scalar product is denoted by $\langle \cdot , \cdot \rangle$, with a subscript to specify the space, if necessary. $L_2(\Xi, K)$ is the space of Hilbert-Schmidt operators from $\Xi$ to $K$, endowed with the Hilbert-Schmidt norm, which makes it a separable Hilbert space.

A cylindrical Wiener process $\{W_t, t \geq 0\}$ defined on a probability space $(\Omega, E, \mathbb{P})$ with values in a Hilbert space $\Xi$, a family of linear mappings $\{\xi_n\}$ from $\Xi$ to $L^2(\Omega)$ on $\{W_t, t \geq 0\}$, is denoted by $\xi \mapsto \langle \xi, W_t \rangle$, such that

(i) for every $\xi \in \Xi$, $\langle \xi, W_t \rangle$, $t \geq 0$ is a real (continuous) Wiener process;

(ii) for every $\xi_1, \xi_2 \in \Xi$ and $t \geq 0$, $E(\langle \xi_1, W_t \rangle \cdot \langle \xi_2, W_t \rangle) = \langle \xi_1, \xi_2 \rangle \Delta t$.

Let $(\mathcal{F}_t)_{t \geq 0}$ denote the natural filtration of $W$, except section 3, augmented with the family of $\mathbb{P}$-null sets. The filtration $(\mathcal{F}_t)$ satisfies the usual conditions. All the concepts of measurability for stochastic processes (e.g. predictability etc.) refer to this filtration. By $\mathcal{P}$ we denote the predictable $\sigma$-algebra and by $\mathcal{B}(\Lambda)$ the Borel $\sigma$-algebra of any topological space $\Lambda$. 

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Next we need to define some classes of stochastic processes with values in a Hilbert space $K$.

- $L^p_p(\Omega; L^2(0, \zeta; K))$ defined for $\zeta \in (0, +\infty]$ and $p \in [1, \infty)$, denotes the space of equivalence classes of processes $Y : \Omega \times [0, \zeta) \to K$, admitting a predictable version and such that

$$|Y|^p_{L^p_p(\Omega; L^2(0, \zeta; K))} = \mathbb{E}\left(\int_0^\zeta |Y_t|^2_K \, dt\right)^{p/2}.$$

Elements of $L^p_p(\Omega; L^2(0, \zeta; K))$ are identified up to modification.

- $L^p_p(\Omega; C(0, \zeta; K))$, defined for $\zeta \in (0, +\infty)$ and $p \in [1, \infty)$, denotes the space of predictable processes $\{Y_t, t \in [0, \zeta]\}$ with continuous paths in $K$ and norms

$$|Y|^p_{L^p_p(\Omega; C(0, \zeta; K))} = \mathbb{E}\sup_{t \in [0, \zeta]} |Y_t|^p_K < \infty.$$

Elements of $L^p_p(\Omega; C(0, \zeta; K))$ are identified up to indistinguishability.

Moreover we give the notations $L^p_{p,loc}(\Omega; L^2(0, \infty; K))$ and $L^p_{p,loc}(\Omega; C(0, \infty; K))$ by their obvious meaning of locally finite $p$-th norms.

In addition, we recall notations and basic facts on a class of differentiable maps acting among Banach spaces, especially those suitable for our purposes. We notice that Gâteaux differentiability is more appropriate than Fre’chet differentiability when dealing with the continuity of the Gâteaux derivatives ([2], [3], and [4]).

Let $X, Z, V$ denote Banach spaces. We say that a mapping $F : X \to V$ belongs to the class $G^1(X, V)$ if it is continuous, Gâteaux differentiable on $X$, and its Gâteaux derivative $\nabla F : X \to L(X, V)$ is strongly continuous. The last condition is equivalent to the fact that for every $h \in X$ the map $\nabla F(\cdot)h : X \to V$ is continuous. Next, for a function $F : X \times Y \to V$, by $\nabla_x F(x, y)$ we denote the partial Gâteaux derivative with respect to the first argument ([2] and [4]). Similarly, $F : X \times Y \to V$ belongs to the class $G^{1,0}(X \times Y, V)$ if it is continuous, Gâteaux differentiable with respect to $x$ on $X \times Y$, and $\nabla_x F : X \times Y \to L(X, V)$ is strongly continuous.

Note that in [2], $\nabla F : X \to L(X, V)$ is not continuous in general if $L(X, V)$ is endowed with the norm operator topology; apparently, if this happens then $F$ is Fre’chet differentiable on $X$. It
can be verified that if $F \in \mathcal{G}^1 (X, V)$ then $(x, h) \rightarrow \nabla F(x)h$ is continuous from $X \times X$ to $V$; if, in addition, $G$ is in $\mathcal{G}^1 (V, Z)$ then $G(F)$ belongs to $\mathcal{G}^1 (X, Z)$ and the chain rule holds: $\nabla (G(F))(x) = \nabla G(F(x)) \nabla F(x)$.

2.2. Definitions and Hypotheses.

Here we need to introduce several definitions and assumptions from [1], [2], [3], and [4]. Let us consider the forward equation of (1.4)

$$X_t = e^{tA}x + \int_0^t e^{(t-s)A} F(X_s)ds + \int_0^t e^{(t-s)A} G(X_s)dW_s, \ t \geq 0. \ (2.1)$$

Under assumption 4.1 in [2], we denote $\{X_t(x), t \geq 0\}$ as the solution. The transition semi-group $(P_t)_{t \geq 0}$ associated to the process $\{X_t\}$ is defined in the usual way:

$$P_t[\varphi](x) = \mathbb{E} \varphi(X_t(x)), \ x \in H. \ (2.2)$$

for each bounded measurable function $\varphi : H \rightarrow \mathbb{R}$. Generally, the generator $L$ of $(P_t)$ is the operator satisfying

$$L\varphi(x) = \frac{1}{2} \text{Trace}(G(x)G(x)^* \nabla^2 \varphi(x)) + \langle Ax + F(x), \nabla \varphi(x) \rangle. \ (2.3)$$

The solvability of the following nonlinear stationary Kolmogorov equation will be studied in section 3,

$$Lu(x) - \lambda u(x) + \psi(x, u(x), \nabla u(x)G(x)) = 0, \ x \in H, \ (2.4)$$

where the function $\psi : H \times \mathbb{R} \times \Xi^* \rightarrow \mathbb{R}$ satisfies appropriate conditions in [2] and [3], and $\lambda$ is a real given number. Note that, for $x \in H$, $\nabla u(x)$ belongs to $H^*$, therefore $\nabla u(x)G(x)$ is in $\Xi^*$.

We also need the following definition of a mild solution ([2] and [3]).

**Definition 2.1** We say that a function $u : H \rightarrow \mathbb{R}$ is a mild solution of the nonlinear stationary Kolmogorov equation (2.4) if the following conditions hold:

(i) $u \in \mathcal{G}^1 (H, \mathbb{R})$ and for some constant $C > 0$, we have $|u(x)| \leq C$, $|\nabla_x u(x)h| \leq C|h|$, for all $x \in H, h \in H$.

(ii) The following equality holds, for every $x \in H$ and $T \geq 0$:

$$u(x) = e^{-\lambda T} P_T [u](x) + \int_0^T e^{-\lambda t} P_t [\psi(\cdot, u(\cdot, \nabla u(\cdot)G(\cdot)))](x) dt. \ (2.5)$$
Together with (2.1), the backward equation of (1.4) can be expressed as

$$ Y_t - Y_T + \int_t^T Z_s \, dW_s + \lambda \int_t^T Y_s \, ds = - \int_t^T \psi (X_s, Y_s, Z_s) \, ds, \quad 0 \leq t \leq T < \infty, $$

(2.6)

where $\psi : H \times R \times \Xi^* \to R$ and $\lambda$ are the same that stated in equation (2.4).

There are still some assumptions needed from [3], which may lead the existence and uniqueness of a mild solution for (1.1) associated with (1.4). However, our main goal is to characterize the mild solution for (1.1), hence, they will be just listed as following.

**Hypothesis 2.1** (i) The operator $A$ is the generator of a strongly continuous semigroup $e^{\lambda t}$, $t \geq 0$, in a Hilbert space $H$. We denote by $m$ and two constants such that $|tAe|$ $\leq m$ $\epsilon t$, for all $t \geq 0$.

(ii) $F : H \to H$ satisfies, for some constant $L > 0$,

$$ |F(x) - F(y)| \leq L|x - y|, \quad x, y \in H. $$

(iii) $G$ denotes a mapping from $H$ to $L(H)$, $\Xi^*$ such that for every $\xi \in \Xi$ the map $G(\cdot)\xi : H \to H$ is measurable, $e^{\lambda t}G(x) \in L(\Xi, H)$ for every $t > 0$ and $x \in H$, and

$$ |e^{\lambda t}G(x)|_{L(\Xi, H)} \leq Lt^{-\beta} e^{\lambda t} (1 + |x|), $$

$$ |e^{\lambda t}G(x) - e^{\lambda t}G(y)|_{L(\Xi, H)} \leq Lt^{-\beta} e^{\lambda t} |x - y|, \quad t > 0, x, y \in H, $$

for some constants $L > 0$ and $\beta \in [0, 1/2)$.

**Hypothesis 2.2** (i) $\psi$ is uniformly Lipschitz in $z$ with Lipschitz constant $K$, that is

$$ |\psi(x, y, z) - \psi(x, y, z')| \leq K|z - z'|, \quad \forall x \in H, \forall y \in R, \forall z, z' \in \Xi^*. $$

(ii) $(x, y) \to \psi(x, y, z)$ is continuous for all $z \in \Xi^*$.

(iii) There exists a continuous and increasing function $\theta : R_+ \to R_+$, such that

$$ |\psi(x, y, z)| \leq |\psi(x, 0, z)| + \theta(|y|), \quad \forall x \in H, \forall y \in R, \forall z \in \Xi^*. $$

(iv) $\lambda > 0$ and $\psi$ is monotone in $y$ in the following sense:

$$ \forall (x, y, y', z), (y - y')(\psi(x, y, z) - \psi(x, y', z)) \leq 0. $$

(v) $\sup_{x \in H} |\psi(x, 0, 0)| := M < +\infty$.

**Hypothesis 2.3** (i) $G$ does not depend on $x$ (that is $G \in L(\Xi, H)$ with $|e^{\lambda t}G(x)|_{L(\Xi, H)} \leq Lt^{-\beta} e^{\lambda t}$, for a suitable $\beta \in [0, 1/2)$).

(ii) We have $F(\cdot) \in G^1(H, H)$.

(iii) Operators $A + F_x(x)$ are dissipative (that is $\langle Ay, y \rangle + \langle F_x(x)y, y \rangle \leq 0$ for all $x \in H$ and $y \in$...
\[ \psi(\cdot, \cdot, \cdot) \in C^1(H \times \mathbb{R} \times \Xi^*, \mathbb{R}) \text{ and } |\nabla_x \psi(x, y, z)|_{H^*} \leq c, |\nabla_y \psi(x, y, z)| \leq c, |\nabla_z \psi(x, y, z)| \leq c, \]

for a suitable constant \( c > 0 \) and all \( x \in H, y \in \mathbb{R}, z \in \Xi^* \).

(v) \( \nabla_y \psi(x, y, z) \leq 0 \).

**Hypothesis 2.4**

(i) \( |G(x)|_{L(H, H^*)} \leq K, \forall x \in H \) and for a suitable constant \( K \).

(ii) \( G(x) \) has a bounded inverse: \( \exists \ B \text{ such that } |G(x)^{-1}|_{L(H, H)} \leq B, \forall x \in H \).

(iii) \( F(\cdot) \in C^1(H, H) \) and for every \( t > 0 \), \( e^{tG} \in C^0(H, L_2(\Xi, H)) \).

(iv) \( \psi \) is uniformly Lipschitz in \( y \) with Lipschitz constant \( K \), namely: \( \forall x \in H, \forall y, y' \in \mathbb{R}, \forall z \in \Xi^*, |\psi(x, y, z) - \psi(x, y', z)| \leq K|y - y'| \).

We need one more important Theorem ([2] and [3]) which will be useful in section 3 and only be listed as below without proof.

**Theorem 2.1** Assume that Hypothesis 2.1, Hypothesis 2.2, and either Hypothesis 2.3 or Hypothesis 2.4 hold then equation (1.1) has a unique mild solution given by the formula

\[ u(x) = Y_0^x. \] (2.6)

Furthermore the following holds:

\[ Y_t^x = u(X_t^x), \quad Z_t^x = \nabla u(X_t^x)G(X_t^x). \] (2.7)

3. Main Results.

We wish to apply the conditions and results in the previous section to perform the synthesis of the optimal control for a general nonlinear control system on an infinite time horizon. To be able to use non-smooth feedbacks we put the problem in the framework of weak control. Likewise, we follow the steps in [2] and [3] with appropriate modifications.

As before by \( H, \Xi \) we denote separable real Hilbert spaces and by \( U \) we denote a Banach space.

For fixed \( x_0 \in H \) an admissible control system (a.c.s) is given by \((\Omega, \mathcal{F}, (F_t)_{t \geq 0}, \mathbb{P}) \), \( \{W_t, t \geq 0\} \) where

- \((\Omega, \mathcal{F}, \mathbb{P})\) is a complete probability space and \((F_t)_{t \geq 0}\) is a filtration on it satisfying the usual conditions.
\[ \{ W_t : t \geq 0 \} \text{ is a } \mathbb{F}-\text{valued cylindrical Wiener process with regard to the filtration } (\mathcal{F}_t) \text{ and the probability } \mathbb{P}. \]

\[ u : \Omega \times [0, \infty) \to \mathcal{U} \text{ is a predictable process (with regard to } (\mathcal{F}_t)_{t \geq 0}) \text{ that satisfies the constraint: } u_t \in \mathcal{U}, \mathbb{P}\text{-a.s. for a.e. } t \geq 0, \text{ where } \mathcal{U} \text{ is a fixed closed subset of } U. \]

To each a.c.s. we associate the mild solution \( X^u \in L^p_p(\Omega; C(0, T; H)) \) (for arbitrary \( T > 0 \) and arbitrary \( r \geq 1 \)) with the state equation:

\[
\begin{align*}
\begin{cases}
dX_t^u &= [AX_t^u + F(X_t^u) + G(X_t^u)\gamma(X_t^u)]dt + G(X_t^u)dW_t, \quad t \geq 0, \\
X_0^u &= x_0 \in H,
\end{cases}
\end{align*}
\]

and the cost functional:

\[
J(x_0, u) = \mathbb{E} \int_0^\infty e^{-s} g(X_s^u, u_s) ds,
\]

where \( g : H \times \mathcal{U} \to \mathbb{R} \). Our purpose is to minimize the functional \( J \) over all a.c.s. Notice that there are more technical assumptions, such as Hypotheses 2.1 to 2.4, on the operator \( G \) in the control term in order to obtain the existence and uniqueness of the solution of equation (1.1).

We define in a classical way the Hamiltonian function related to the above problem: for all \( x \in H, p \in \mathcal{U}^* \),

\[ \psi_0(x, p) = \inf_{u \in \mathcal{U}} \{ g(x, u) + p\gamma(u) \} \]

and

\[ \Gamma(x, p) = \{ u \in \mathcal{U} : g(x, u) + p\gamma(u) = \psi_0(x, p) \}. \]

We will work under the following additional assumption.

**Hypothesis 3.1** The following holds:

1. \( A, F, G, \) and \( \gamma \) satisfy Hypothesis 2.1, Hypothesis 2.2, and either Hypothesis 2.3 or Hypothesis 2.4.
2. The map \( R : H \to L(\mathcal{U}, \mathbb{F}) \).
3. \( \gamma : \mathcal{U} \to \mathcal{U} \) is bounded.
4. \( g : H \times \mathcal{U} \to \mathbb{R} \) is continuous and bounded.

The bounded assumptions on \( g \) and \( \gamma \) prevent us from further constraints on mapping \( R \). Otherwise, it must need further complicated conditions for the mapping \( R \) to guarantee the
existence and uniqueness of the mild solution of (1.1) (see [2] and [3]). We also define \( \psi(x, z) = -\psi_0(x, zR(x)) \), \( x \in H, z \in \Xi^* \).

Then we notice that for all \( \lambda > 0 \) the cost functional is well defined and \( J(x_0, u) < \infty \) for all \( x_0 \in H \) and all a.c.s.

For all \( \lambda > 0 \) the stationary Hamilton-Jacobi-Bellman equation relative to the above stated problem can be rewritten as:

\[
\mathcal{L}v(x) = \lambda v(x) - \psi(x, \nabla v(x)G(x)R(x)), \ x \in H,
\]

which allows a unique mild solution and \( \mathcal{L} \) is the generator stated in equation (2.3).

**Theorem 3.1** Assume Hypothesis 3.1 and suppose that \( \lambda > 0 \). Then the following holds

1. For all a.c.s. we have \( J(x_0, u) \geq v(x_0) \).
2. The equality holds if and only if the following feedback law is verified by \( u \) and \( X^u \):
   \[
u \in \Gamma(X^u, \nabla v(X^u))G(X^u)R(X^u), \ \text{\( \mathcal{P} \)-a.s. for a.e. } t \geq 0.
   \]
3. If for all \( x \in H \) and \( z \in \Xi^* \) and \( \Gamma_0 : H \times \Xi^* \to \mathcal{U} \), \( \Gamma_0(x, z) \) is a measurable selection of \( \Gamma \) then there exists an a.c.s. in the system of the closed loop equation

\[
d\bar{X}_t = A \bar{X}_t dt + G(\bar{X}_t)R(\bar{X}_t)\gamma(\Gamma_0(\bar{X}_t, \nabla v(\bar{X}_t))G(\bar{X}_t)R(\bar{X}_t))dt \\
\quad + F(\bar{X}_t) dt + G(\bar{X}_t) dW_t, \ t \geq 0,
\]

\[
\bar{X}_0 = x_0 \in H,
\]

which admits a solution. Furthermore, if we set \( \bar{u}_t = \Gamma_0(\bar{X}_t, \nabla v(\bar{X}_t))G(\bar{X}_t)R(\bar{X}_t) \) then the couple \((\bar{u}, \bar{X})\) is optimal for the control problem.

**Proof.** We follow the similar approaches in [3] by denoting \( \rho(T) \) the Girsanov density

\[
\rho(T) = \exp\left(-\int_0^T (R(X_s)\gamma(u_s), dW_s) - \frac{1}{2} \int_0^T |R(X_s)\gamma(u_s)|^2 ds \right)
\]

and let \( \bar{\mathcal{P}}_T \) be the probability measure on \( \mathcal{F}_t \) defined by \( \bar{\mathcal{P}}_T = \rho(T) \mathcal{P}_{|\mathcal{F}_t} \) and let \( \bar{E}_T \) be the corresponding expectation. We notice that under \( \bar{\mathcal{P}}_T \) the process

\[
\bar{W}_t = W_t + \int_0^t R(X_s)\gamma(u_s)ds
\]

is a cylindrical Wiener process. Compared with \( \bar{W} \), equation (3.1) can be written:
Let $v$ be the unique mild solution of equation (3.2). Consider the following finite horizon Markovian forward-backward system (with respect to probability $\bar{P}_T$ and to the filtration generated by $\{\bar{W}_t : t \in [0, T]\}$)

\[
\begin{cases}
\tilde{X}_t = e^{t\lambda}x + \int_0^t e^{(t-s)\lambda} F(\tilde{X}_s(x)) ds + \int_0^t e^{(t-s)\lambda} G(\tilde{X}_s(x)) d\tilde{W}_s, \ t \geq 0, \\
\tilde{Y}_t = -v(\tilde{X}_T(x)) + \int_t^T \tilde{Z}_s(x)d\tilde{W}_s + \lambda \int_t^T \tilde{Y}_s(x) ds = \int_t^T \psi(\tilde{X}_s(x), \tilde{Z}_s(x)) ds, \ 0 \leq t \leq T < \infty.
\end{cases}
\] (3.6)

Let $(\tilde{X}(x), \tilde{Y}(x), \tilde{Z}(x))$ be its unique solution with the three processes predictable relative to the filtration generated by $\{\bar{W}_t : t \in [0, T]\}$, such that $\bar{P}_T(\sup_{t \in [0, T]} |\tilde{X}_t|) < +\infty$, $\tilde{Y}(x)$ bounded and continuous, $\bar{E}_T \int_0^T |\tilde{Z}_t|^2 dt < +\infty$. Moreover, by Theorem 2.1 and uniqueness of the solution of system (3.6) (see [1]), we have that

\[
\tilde{Y}_t(x) = v(\tilde{X}_t(x)), \quad \tilde{Z}_t(x) = \nabla v(\tilde{X}_t(x))G(\tilde{X}_t(x))R(\tilde{X}_t(x)).
\] (3.7)

Comparing the forward equation in (3.6) with the state equation, rewritten as (3.5), and choosing $x = x_0$ we get $\tilde{X}_t(x_0) = X_t^w, \ t \in [0, T], \bar{P}$-a.s. Since by the second equation of (3.6) we have

\[
d\tilde{Y}_t = \tilde{Z}_t(x)d\tilde{W} + \lambda \tilde{Y}_t dt - \psi(\tilde{X}_t(x), \tilde{Z}_t(x)) dt.
\]

This implies that

\[
e^{-\lambda t}d\tilde{Y}_t = e^{-\lambda t}\tilde{Z}_t(x)d\tilde{W} + \lambda e^{-\lambda t}\tilde{Y}_t dt - e^{-\lambda t}\psi(\tilde{X}_t(x), \tilde{Z}_t(x)) dt.
\]

Then using integration by parts to obtain

\[
e^{-\lambda t} \tilde{Y}_t - e^{-\lambda T}v(\tilde{X}_T(x)) + \int_t^T e^{-\lambda s} \tilde{Z}_s(x)d\tilde{W}_s + \lambda \int_t^T e^{-\lambda s} \tilde{Y}_s(x) ds = \int_t^T e^{-\lambda s}\psi(\tilde{X}_s(x), \tilde{Z}_s(x)) ds.
\]
Taking $t = 0$ and putting back the original noise $W$, we get
\[
\tilde{Y}_0(x_0) + \int_0^T e^{\lambda s} \tilde{Z}_s(x_0) dW_s \\
= \int_0^T e^{\lambda s} [\psi(X^u_s, \tilde{Z}_s(x_0)) - \tilde{Z}_s(x_0) R(X^u_s) \gamma(u_s)] ds + e^{-\lambda T} \nu(\tilde{X}_T(x_0)). \tag{3.8}
\]

Using the identification in (3.7) and taking expectation on (3.8) with respect to $\mathbb{P}$, we yield
\[
e^{-\lambda T} \mathbb{E} [\nu(\tilde{X}_T(x_0)) - \nu(x_0)] = -\mathbb{E} \int_0^T e^{\lambda s} \psi(X^u_s, \nabla v(X^u_s) G(X^u_s) R(X^u_s)) ds \\
+ \mathbb{E} \int_0^T e^{\lambda s} \nabla v(X^u_s) G(X^u_s) R(X^u_s) \gamma(u_s) ds.
\]

Recalling that $\nu$ is bounded, letting $T \to \infty$ then adding and subtracting an $\mathbb{E} \int_0^\infty e^{-\lambda s} g(X^u_s, u_s) ds$, we conclude
\[
J(x_0, u) = \nu(x_0) + \mathbb{E} \int_0^\infty e^{\lambda s} [-\psi(X^u_s, \nabla v(X^u_s) G(X^u_s) R(X^u_s)) \\
+ \nabla v(X^u_s) G(X^u_s) R(X^u_s) \gamma(u_s) + g(X^u_s, u_s)] ds.
\]

The above equality immediately implies that $\nu(x_0) \leq J(x_0, u)$. The equality holds if and only if (3.3) holds. In order to prove the existence of a weak solution to (3.4), we apply a consequence of Girsanov Theorem in [7] and follow the similar approaches in [2]. That is, let $X \in L^p_{\mathbb{P}, \text{loc}}(\Omega, C(0, +\infty; H))$ be the mild solution of
\[
dX_t = AX_t dt + F(X_t) dt + G(X_t) dW_t \\
X_0 = x_0
\tag{3.9}
\]
and let $\hat{\mathbb{P}}$ be the probability on $\Omega$ under which
\[
\hat{W}_t := W_t - \int_0^t R(X_s) \gamma(\Gamma_0(X_s, \nabla v(X_s) G(X_s) R(X_s))) ds.
\]
is a Wiener process. Then $X_t$ is the mild solution of (3.4) related to the the probability $\hat{\mathbb{P}}$ and $\hat{W}$. If we rewrite (3.9) in terms of $\{\hat{W}_t : t \geq 0\}$, we get
\[
dX_t = AX_t dt + F(X_t) dt + G(X_t) [R(X_t) \gamma(\Gamma_0(X_t, \nabla v(X_t) G(X_t) R(X_t))) dt + d\hat{W}_t],
\]

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\[ X_0 = x_0. \]

Hence, conclude the proof.

**Remark 3.1.** The existence of a weak solution to (3.4) is also proved by [3] with a more strict approach. However, our purpose is only to characterize an optimal control process \( u \) that minimizes an infinite time horizon cost functional hence we follow a less strict approach in [2].

**Remark 3.2.** The coefficient of the drift term \( R(X_t) \) is also a crucial factor on the characterization of an optimal control \( u \), though there is no strict assumptions on \( R \).

4. Examples.

**Example 4.1** (Theorem 5.1, [3]) Under Hypothesis 3.1, suppose that \( \lambda > 0 \) and the map \( R_0 : U \rightarrow \Xi \) is bounded. Then the following holds

1. For all a.c.s. we have \( J(x_0, u) \geq \nu(x_0) \).
2. The equality holds if and only if the following feedback law is verified by \( u \) and \( X^u \):
   \[ u_t \in \Gamma(X^u_t, \nabla \nu(X^u_t)G(X^u_t)), \text{ P-a.s. for } a.e. \ t \geq 0. \] (4.1)
3. If for all \( x \in H \) and \( z \in \Xi^* \) and \( \delta : H \times \Xi^* \rightarrow 'U \), \( \delta(x, z) \) is a measurable selection of \( \Gamma \) then there exists an a.c.s in the system of the closed loop equation
   \[
   \begin{cases}
   d\overline{X}_t = A\overline{X}_t dt + G(\overline{X}_t)R_0(\delta(\overline{X}_t, \nabla \nu(\overline{X}_t)G(\overline{X}_t)))dt + F(\overline{X}_t)dt + G(\overline{X}_t)dW_t, & t \geq 0, \\
   \overline{X}_0 = x_0 \in H,
   \end{cases}
   \] (4.2)

which admits a solution. Besides, if we set \( \pi_t = \delta(\overline{X}_t, \nabla \nu(\overline{X}_t)G(\overline{X}_t)) \) then the couple \( (\pi, \overline{X}) \) is optimal for the control problem.

**Proof.** Here taking \( R(X^u_t) = 1 \) and \( \gamma(u_t) = R_0(u_t) \) in (3.1) then we can get the desired results by using Theorem 3.1.

Before discussing the second example, we need to formulate the stochastic optimal control problem in the strong sense. Consider that (i) a bounded linear operator \( Q : H \rightarrow H \) is positive and diagonal with respect to the basis \( \{ e_k \}_{k \in \mathbb{N}} \), with eigenvalues \( \{ \lambda_k \}_{k \in \mathbb{N}} \); (ii) \( W \) is a cylindrical Wiener process in \( H \) and \( Q \) is its covariance operator. Let mapping \( R_0 : U \rightarrow H \) and refer to equations (2.1) and (2.6), then consider the forward-backward system

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\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{dX^u_t}{dt} = [AX^u_t + \sqrt{Q} R^u_t \mu] dt + \sqrt{Q} dW^u_t, \quad 0 \leq t,
X^u_0 = x,
\end{array} \right.
\end{align*}
\]

(4.3)

Here the solution of this equation is denoted by \( X^u_{t,x} \) or simply by \( X^u_t \).

The special structure of (4.3) inspires the study of the related optimal control problem by means of FBSDEs. System (4.3) leads to a semi-linear Hamilton Jacobi Bellman equation with the structure of the elliptic PDE instead of parabolic PDE in [5], studied in the previous sections, which may be expressed as follow.

\[
L v(x) = \lambda v(x) + \psi(x, \nabla \sqrt{Q} v(t, x)) + l(x), \quad x \in H,
\]

where the second order operator \( L \) is:

\[
L \varphi(x) = \frac{1}{2} \text{Trace} (Q^\ast \nabla^2 \varphi(x)) + \langle Ax, \nabla \varphi(x) \rangle,
\]

\( \nabla \sqrt{Q} \) is the \( \sqrt{Q} \)-directional derivative, and \( \nabla \sqrt{Q} v(t, x) \) is bounded and continuous by [5].

Likewise the operator \( G(x) \) in the previous sections, we need more technical restrictions on the operator \( \sqrt{Q} \) in the control term. On the contrary the presence of the operator \( R \) allows more generality.

Let us define the cost

\[
J(t; x; u) = E \int_0^\infty e^{-\lambda t} [h(u_s) + l(X^u_s)] ds
\]

for real functions \( l \) on \( H \) and \( h \) on \( U \). The control problem in strong formulation is to minimize this functional \( J \) over all admissible controls \( u \).

By \( \mathcal{A}_u \) we denote the set of admissible controls \( u_t \) and the \( U \)-valued predictable processes \( \{ u_t \} \) satisfy

\[
E \int_0^T |u_t|^q dt < \infty.
\]

This summability requirement is verified by the boundedness of \( h \): a control process which is not \( q \)-summable would have infinite cost. We denote by \( J(t; x) = \inf_{u \in \mathcal{A}_u} J(t; x; u) \) the value function of
this problem and, if it exists, by \( \bar{u} \) the control minimizing the cost function is called optimal control.

**Example 4.2** (Theorem 3.12, [5]) Suppose that (i) \( h : U \to \mathbb{R} \) and \( l : H \to \mathbb{R} \) are bounded and continuous, (ii) \( R_0 : U \to H \) is measurable and \( |R_0(u)| \leq C, \exists C > 0 \), for every \( u \in U \), (iii) \( \delta : H \to U \) is a measurable selection of \( \Gamma \) which is defined by \( \Gamma(z) = \{ u \in U : z R_0(u) + h(u) = \psi(z) \} \), and (iv) the Hamiltonian function \( \psi \) satisfies Gâteaux differentiability assumptions stated in [5]. For every \( t \in [0; \infty) \), \( x \in H \) and for all admissible control systems we have \( J(t; x; u(\cdot)) \geq \nu(t; x) \), and the equality holds if and only if

\[
u = \Gamma(\nabla \psi(s, X^u_s(x)) \sqrt{Q}) = \Gamma(\nabla \sqrt{Q} \nu(s, X^u_s(x))).\]

Besides, the closed loop equation

\[
dX^u_s = \{AX^u_s + \sqrt{Q} R_0[\delta(\nabla \sqrt{Q} \nu(t, X^u_s))]\}ds + \sqrt{Q}dW_s, \quad s \geq 0,
\]

admits a weak solution \((\Omega; \mathcal{F}; (\mathcal{F}_t)_{t \geq 0}; \mathbb{P}; W; X)\) which is unique in law. Furthermore, if we set \( u_s = \delta(\nabla \psi(s, X^u_s(x)) \sqrt{Q}) = \delta(\nabla \sqrt{Q} \nu(s, X^u_s(x))) \), we obtain an optimal admissible control system \((W; u; X)\).

**Proof.** Since the definition of Gâteaux differentiability yields immediately

\[
\nabla \sqrt{Q} \nu(s, X^u_s(x)) \xi = \nabla \nu(s, X^u_s(x)) \sqrt{Q} \xi, \quad \forall \xi \in H, \text{ then following (3.1), putting } R(X^u_s) = 1, G(X^u_s) = \sqrt{Q}, \gamma(u_s) = R_0(u_s), \text{ and } g(X^u_s, u_s) = h(u_s) + l(X^u_s), \text{we can get the desired results by using Theorem 3.1.}
\]

**Remark 4.1.** We notice that the boundedness requirement of \( R_0 \) is a key condition to admit a unique mild solution.

5. References.


